

# ON FORKING AND DEFINABILITY OF TYPES IN SOME DP-MINIMAL THEORIES

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**Abstract.** We prove in particular that, in a large class of dp-minimal theories including the  $p$ -adics, definable types are dense amongst non-forking types.

**§1. Introduction and preliminaries.** In this short note, we show how the techniques from [5] can be adapted to prove the *density* of definable types in a large class of dp-minimal theories. Density of definable types is the following: for any  $\phi(x)$  which does not fork over a model  $M$ , there is a global type  $p(x)$  definable over  $M$  and containing  $\phi(x)$ . We prove this for dp-minimal  $T$  satisfying an extra property—property (D)—which says that unary definable sets contain a type that is definable over the same parameters as the set. This holds in particular if definable sets have natural *generic* definable types. This also holds whenever  $T$  has definable Skolem functions. In particular our theorem applies to the field  $\mathbb{Q}_p$  of  $p$ -adic numbers.

Throughout,  $T$  is a complete countable theory. We let  $\mathcal{U}$  be a monster model. By a global type, we mean a type over  $\mathcal{U}$ . We write  $M \prec^+ N$  to mean  $M \prec N$  and  $N$  is  $|M|^+$ -saturated.

The notation  $\phi^0$  means  $\neg\phi$  and  $\phi^1$  means  $\phi$ .

If  $M \prec^+ N$  and  $p \in S(N)$ , then  $p$  is  $M$ -invariant if for any  $b, b' \in N$  and any formula  $\phi(x; y)$ ,  $b \equiv_M b'$  implies  $p \vdash \phi(x; b) \leftrightarrow \phi(x; b')$ . Any  $M$ -invariant type over  $N$  extends in a unique way to a global  $M$ -invariant type. Thus there is no harm in considering only global invariant types.

We refer to [5] or to [4] for basic facts about NIP theories, though we will now collect all the statements that we need.

First recall that in an NIP theory, a global type  $p$  does not fork over a model  $M$  if and only if it is  $M$ -invariant.

If  $p(x)$  and  $q(y)$  are two global  $M$ -invariant types, then  $p(x) \otimes q(y)$  denotes the global type  $r(x, y)$  defined as  $\text{tp}(a, b/\mathcal{U})$  where  $b \models q$  and  $a \models p|_{\mathcal{U}a}$  (invariant extension of  $p$  to  $\mathcal{U}a$ ).

If  $p(x) \otimes q(y) = q(y) \otimes p(x)$ , then we say that  $p$  and  $q$  *commute*. It is not hard to see that, in any theory  $T$ , a global  $M$ -invariant type is definable if and only if it commutes with all global types finitely satisfiable in  $M$  (see [5, Lemma 2.3]).

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Next we recall the notion of strict non-forking from [1]. Let  $M$  be a model of an NIP theory. A sequence  $(b_i)_{i < \omega}$  is strictly non-forking over  $M$  if for each  $i < \omega$ ,  $\text{tp}(b_i/b_{<i}M)$  is strictly non-forking over  $M$  which means that it extends to a global type  $\text{tp}(b_*/\mathcal{U})$  such that both  $\text{tp}(b_*/\mathcal{U})$  and  $\text{tp}(\mathcal{U}/Mb_*)$  are non-forking over  $M$ . We will only need to know two facts about strict non-forking sequences (both proved in [1], see also [4, Chapter 5]):

(Existence) Given  $b \in \mathcal{U}$  and  $M \models T$ , there is an indiscernible sequence  $b = b_0, b_1, \dots$  which is strictly non-forking over  $M$ . We call such a sequence a *strict Morley sequence* of  $\text{tp}(b/M)$ .

(Witnessing property) If the formula  $\phi(x; b)$  forks over  $M$ , then for any strictly non-forking indiscernible sequence  $b = b_0, b_1, \dots$ , the type  $\{\phi(x; b_i) : i < \omega\}$  is inconsistent.

If  $\phi(x; y)$  is an NIP formula, we let  $\text{alt}(\phi)$  be the *alternation number* of  $\phi$ , namely the maximal  $n$  for which there is an indiscernible sequence  $(b_i : i < \omega)$  and a tuple  $a$  with  $\neg(\phi(a; b_i) \leftrightarrow \phi(a; b_{i+1}))$  for all  $i < n$ . If  $(b_i : i < \omega)$  is indiscernible and  $\{\phi(x; b_i) : i < \text{alt}(\phi)/2 + 1\}$  is consistent, then  $\{\phi(x; b_i) : i < \omega\}$  is also consistent.

We will also need the notion of “ $b$ -forking” as defined in Cotter and Starchenko’s paper [2] and as recalled in [5]. For this, we assume that  $T$  is NIP.

Assume we have  $M \prec^+ N$  and  $b \in \mathcal{U}$  such that  $\text{tp}(b/N)$  is  $M$ -invariant. We say that a formula  $\psi(x, b; d) \in L(Nb)$   $b$ -divides over  $M$  if there is an  $M$ -indiscernible sequence  $(d_i : i < \omega)$  inside  $N$  with  $d_0 = d$  and  $\{\psi(x, b; d_i) : i < \omega\}$  is inconsistent. We define  $b$ -forking in the natural way.

FACT 1. ( *$T$  is NIP*) *Notations being as above, the following are equivalent:*

- (i)  $\psi(x, b; d)$  does not  $b$ -divide over  $M$ ;
- (ii)  $\psi(x, b; d)$  does not  $b$ -fork over  $M$ ;
- (iii) if  $(d_i : i < \omega)$  is a strict Morley sequence of  $\text{tp}(d/M)$  inside  $N$ , then  $\{\psi(x, b; d_i) : i < \omega\}$  is consistent;
- (iii)' if  $(d_i : i < \omega)$  is a strict Morley sequence of  $\text{tp}(d/M)$  inside  $N$ , then  $\{\psi(x, b; d_i) : i < m\}$  is consistent where  $m$  is greater than the alternation number of  $\psi(x, y; z)$ ;
- (iv) there is  $a \models \psi(x, b; d)$  such that  $\text{tp}(a, b/N)$  is  $M$ -invariant.

Finally a theory  $T$  is dp-minimal if for every  $A \subset \mathcal{U}$ , every singleton  $a$  and any two infinite sequences  $I_0, I_1$  of tuples, if  $I_k$  is indiscernible over  $AI_{1-k}$ ,  $k = 0, 1$ , then for some  $k \in \{0, 1\}$ ,  $I_k$  is indiscernible over  $Aa$ .

Any o-minimal or weakly o-minimal theory is dp-minimal, as is the theory of the fields of  $p$ -adics.

The following theorem was proved in [5]:

THEOREM 2. ( *$T$  is dp-minimal*) *Let  $p(x)$  be a global  $M$ -invariant type in a single variable, then  $p$  is either definable over  $M$  or finitely satisfiable in  $M$ .*

**§2. The main theorem.** We will say that  $T$  has property (D) if for every set  $A$  (of real elements) and consistent formula  $\phi(x) \in L(A)$ , with  $x$  a single variable, there is an  $A$ -definable complete type  $p \in S_x(A)$  extending  $\phi(x)$ .

We emphasise that the type  $p$  might not extend to a global  $A$ -definable type.

LEMMA 3. *Let  $M \prec N$  and  $b \in \mathcal{U}$  such that  $\text{tp}(b/N)$  is  $M$ -definable. Assume that  $p \in S_x(Mb)$  is a complete  $Mb$ -definable type, then  $p$  extends to a complete type  $q \in S_x(Nb)$  which is  $Mb$ -definable using the same definition scheme as  $p$ .*

PROOF. For each formula  $\phi(x; y, b) \in L(b)$ , there is by hypothesis a formula  $d\phi(y; b) \in L(M)$  such that for every  $d \in M^{|y|}$  we have  $p \vdash \phi(x; d, b)$  if and only if  $\mathcal{U} \models d\phi(d; b)$ . We have to check that the scheme  $\phi(x; y, b) \mapsto d\phi(y; b)$  defines a consistent complete type over  $Nb$ . This follows at once from the fact that  $\text{tp}(b/N)$  is an heir of  $\text{tp}(b/M)$ . Let us check completeness for example. Assume that there is some  $n \in N$  and formula  $\phi(x; y, b)$  such that  $\mathcal{U} \models \neg d\phi(n; b) \wedge \neg d(\phi^0)(n; b)$ . By the heir property, there must be such a tuple  $n$  in  $M$ , which is a contradiction.  $\dashv$

LEMMA 4. (*T is NIP*) *Let  $M \prec^+ N$ ,  $n < \omega$  and assume that any formula  $\theta(y; d) \in L(N)$  with  $|y| = n$  and non-forking over  $M$  extends to an  $M$ -definable type over  $N$ . Let  $\phi(x, y; d) \in L(N)$  be non-forking over  $M$ , where  $|y| = n$  and  $|x| = 1$ . Then we can find a tuple  $(a, b) \models \phi(x, y; d)$  such that  $\text{tp}(a, b/N)$  is  $M$ -invariant and  $\text{tp}(b/N)$  is definable (over  $M$ ).*

PROOF. Let  $(d_i : i < \omega)$  be a strict Morley sequence of  $\text{tp}(d/M)$  inside  $N$ . Let  $m < \omega$  be greater than the alternation number of  $\phi(x, y; z)$ . As the formula  $\phi(x, y; d)$  does not fork over  $M$ , it extends to a global  $M$ -invariant type  $p$ . Then the conjunction  $\psi(x, y; \bar{d}) = \bigwedge_{i < m} \phi(x, y; d_i)$  is in  $p$ . In particular it is consistent and does not fork over  $M$ . The same is true for  $\theta(y; \bar{d}) = (\exists x)\psi(x, y; \bar{d})$ . By hypothesis, we can find some  $b \in \mathcal{U}$  such that  $\text{tp}(b/N)$  is  $M$ -definable and  $\mathcal{U} \models \theta(b; \bar{d})$ . We claim that the formula  $\phi(x; b, d)$  does not  $b$ -fork over  $M$ . Assume that it did. Then the conjunction  $\bigwedge_{i < m} \phi(x, b; d_i)$  would be inconsistent. But this contradicts the fact that  $\theta(b; \bar{d})$  holds. Hence we may find  $a \in \mathcal{U}$  such that  $\phi(a, b; d)$  holds and  $\text{tp}(a, b/N)$  does not fork over  $M$  (equivalently is  $M$ -invariant).  $\dashv$

THEOREM 5. *Assume that  $T$  is dp-minimal and has property (D). Let  $M \models T$  and  $\phi(x; d) \in L(\mathcal{U})$  be non-forking over  $M$ . Then  $\phi(x; d)$  extends to a complete  $M$ -definable type.*

PROOF. The proof is an adaptation of the argument given for Proposition 2.7 in [5]. We argue by induction on the length of the variable  $x$ .

$|x| = 1$ : Assume that  $|x| = 1$  and take  $p(x)$  a global type extending  $\phi(x; d)$  and non-forking over  $M$ . If  $p$  is definable, we are done. Otherwise, by Theorem 2,  $p$  is finitely satisfiable in  $M$ . This implies that  $\phi(x; d)$  has a solution  $a$  in  $M$ . Then  $\text{tp}(a/\mathcal{U})$  does the job.

Induction: Assume we know the result for  $|x| = n$ , and consider a non-forking formula  $\phi(x_1, x_2; d)$ , where  $|x_2| = n$  and  $|x_1| = 1$ . Let  $N \succ M$  sufficiently saturated, with  $d \in N$ . Using the induction hypothesis and Lemma 4, we can find a tuple  $(a_1, a_2) \models \phi(x_1, x_2; d)$  such that  $\text{tp}(a_1, a_2/N)$  is  $M$ -invariant and  $\text{tp}(a_2/N)$  is definable (over  $M$ ).

If  $p = \text{tp}(a_1, a_2/N)$  is definable we are done. Otherwise, there is some type  $q \in S(N)$  finitely satisfiable in  $M$  such that  $p$  does not commute with  $q$ .

Now let  $c \in \mathcal{U}$  such that  $(a_1 \hat{\ } a_2, c) \models p \otimes q$ . Let  $I$  be a Morley sequence of  $q$  over everything. As  $\text{tp}(a_2/N)$  is definable, it commutes with  $q$ . Therefore the

sequence  $\bar{c} = c + I$  is indiscernible over  $Na_2$ . However, it is not indiscernible over  $Na_1a_2$ . Take some  $M \prec^+ N_1 \prec^+ N$  with  $\text{tp}(N_1/Md)$  finitely satisfiable in  $M$ .

Take  $r \in S(\mathcal{U})$  finitely satisfiable in  $N$ . Let  $b \models r|_{Na_2\bar{c}}$ . Build a Morley sequence  $J$  of  $r$  over everything. Then  $b + J$  is indiscernible over  $Na_2\bar{c}$  and  $\bar{c}$  is indiscernible over  $NbJ$ . As  $\bar{c}$  is not indiscernible over  $Na_1a_2$ , by dp-minimality,  $b + J$  must be indiscernible over  $Na_1a_2$ . Hence  $b \models r|_{Na_1a_2\bar{c}}$ .

We have shown that  $r|_{Na_2\bar{c}} \vdash r|_{Na_1a_2\bar{c}}$ . Let  $l = l_r \in \{0, 1\}$  such that  $r(y) \vdash \phi^l(a_1, a_2; y)$ . Then  $r(y)|_{Na_2\bar{c}} \vdash \phi^l(a_1, a_2; y)$ . By compactness, there is a formula  $\theta_r(y)$  in  $r(y)|_{Na_2\bar{c}}$  which already implies  $\phi^l(a_1, a_2; y)$ . Using compactness of the space of global  $N$ -finitely satisfiable types, we can extract from the family  $(\theta_r(y))_r$  a finite subcover  $\mathcal{C}$ . Let  $\theta_l(y)$  be the disjunction of the formulas in  $\mathcal{C}$  that imply  $\phi^l(a_1, a_2; y)$ . Summing up, we have:

$\mathcal{U} \models \theta_l(y) \rightarrow \phi^l(a_1, a_2; y)$ ,  $l = 0, 1$ , and every type finitely satisfiable in  $N$  satisfies either  $\theta_1(y)$  or  $\theta_2(y)$ . In particular, this is true of any point  $n \in N$ .

Write  $\theta_1(y)$  as  $\theta_1(y; a_2, \bar{c}, e)$  exhibiting all parameters, with  $e \in N$ . By invariance of  $\text{tp}(a_1, a_2, \bar{c}/N)$ , we may assume that  $e \in N_1$  and in particular  $\text{tp}(e/Md)$  is finitely satisfiable in  $M$ .

As  $\text{tp}(\bar{c}/Na_2)$  is finitely satisfiable in  $M$ , there is  $\bar{c}' \in M$  such that:

$$\models \theta_1(d; a_2, \bar{c}', e) \wedge (\exists x)(\forall y)(\theta_1(y; a_2, \bar{c}', e) \rightarrow \phi(x; y)).$$

Next,  $\text{tp}(e/Md)$  is finitely satisfiable in  $M$ . As  $\text{tp}(a_2/N)$  is  $M$ -definable, also  $\text{tp}(e/Mda_2)$  is finitely satisfiable in  $M$  and we may find  $e' \in M$  such that the previous formula holds with  $e$  replaced by  $e'$ .

By property (D), there is some  $Ma_2$ -definable type  $p_1(x_1) \in S(Ma_2)$  containing the formula  $(\forall y)(\theta_1(y; a_2, \bar{c}', e') \rightarrow \phi(x; y))$ . By Lemma 3,  $p_1$  extends to a complete  $Ma_2$ -definable type over  $Na_2$ . Let  $a'_1$  realise that type. Then  $\text{tp}(a'_1, a_2/N)$  is  $M$ -definable and we have  $\models \phi(a'_1, a_2; d)$  as required.  $\dashv$

Theorem 5 was proved for *unpackable VC-minimal theories* by Cotter and Starchenko in [2]. This class contains in particular o-minimal theories (for which the result was established earlier by Dolich [3]) and C-minimal theories with infinite branching. We show now that our result generalises Cotter and Starchenko's and covers some new cases, in particular the field of  $p$ -adics.

LEMMA 6. *Let  $A$  be any set of parameters and  $p(x)$  be a global  $\text{acl}(A)$ -definable type. Then  $p|_A$  is  $A$ -definable.*

PROOF. Take  $\phi(x; y) \in L$  and let  $d\phi(y; a)$ ,  $a \in \text{acl}(A)$ , be the  $\phi$ -definition of  $p$ . Then  $\text{tp}(a/A)$  is isolated by a formula  $\phi(z) \in L(A)$ . Define  $D\phi(y) = (\exists z)\phi(z) \wedge d\phi(y; z)$ . Then  $D\phi(y)$  is a formula over  $A$  and defines the same set on  $A$  as  $d\phi(y)$ .  $\dashv$

PROPOSITION 7. *The following classes of theories have property (D):*

- theories with definable Skolem functions;
- dp-minimal linearly ordered theories;
- unpackable VC-minimal theories.

PROOF. Let  $T$  have definable Skolem functions and take a formula  $\phi(x) \in L(A)$ . Then we can find  $a \in \text{dcl}(A)$  such that  $\models \phi(a)$ , and thus  $\text{tp}(a/A)$  is as required.

Assume now that  $T$  is dp-minimal and that the language contains a binary symbol  $<$  such that  $T \vdash “x < y \text{ defines a linear order}”$ . Let  $\phi(x) \in L(A)$  be a formula with  $|x| = 1$ . If the formula  $\phi(x)$  contains a greatest element, then that element is definable from  $A$ , and we conclude as in the previous case. Otherwise, consider the following partial type over  $\mathcal{U}$ :

$$p_0 = \{a < x : a \in \phi(\mathcal{U})\} \cup \{x < b : \phi(\mathcal{U}) < b\} \cup \{\phi(x)\}.$$

Let  $\mathfrak{P}$  be the set of completions of  $p_0$  over  $\mathcal{U}$ . By Lemma 2.8 from [6], any  $p \in \mathfrak{P}$  is definable over  $M$ . In particular,  $\mathfrak{P}$  is bounded. Since  $\mathfrak{P}$  is  $A$ -invariant (setwise), we conclude that every  $p \in \mathfrak{P}$  is  $\text{acl}^{eq}(A)$ -definable. Let  $p$  be such a type. Then by Lemma 6  $p|_A$  is  $A$ -definable.

Finally, let  $T$  be an unpackable VC-minimal theory. We will use results and terminology from [2]. Let  $\phi(x) \in L(A)$  be a consistent formula with  $|x| = 1$ . We work in  $T^{eq}$ . By the uniqueness of Swiss cheese decomposition, there is a consistent formula  $\theta(x)$  over  $\text{acl}(A)$  that defines a Swiss cheese and  $\models \theta(x) \rightarrow \phi(x)$ . The outer ball  $B$  of  $\theta(x)$  is definable over  $\text{acl}(A)$ . The generic type the interior of  $B$  (see [2, Definition 2.9]) is a global type definable over  $\text{acl}(A)$ . Now use Lemma 6.  $\dashv$

Knowing that the theory of the  $p$ -adics has definable Skolem functions, we obtain the following corollary.

**COROLLARY 8.** *Let  $T = Th(\mathbb{Q}_p)$  and  $M \models T$ , then any formula in  $L(\mathcal{U})$  which does not fork over  $M$  extends to an  $M$ -definable type.*

## REFERENCES

- [1] ARTEM CHERNIKOV and ITAY KAPLAN, *Forking and dividing in  $NTP_2$  theories*, *The Journal of Symbolic Logic*, vol. 77 (2012), no. 1, pp. 1–20.
- [2] SARAH COTTER and SERGEI STARCHENKO, *Forking in VC-minimal theories*, *The Journal of Symbolic Logic*, vol. 77 (2012), no. 4, pp. 1257–1271.
- [3] ALFRED DOLICH, *Forking and independence in o-minimal theories*, *The Journal of Symbolic Logic*, vol. 69 (2004), no. 1, pp. 215–240.
- [4] PIERRE SIMON, *A guide to nip theories*, to appear.
- [5] ———, *Invariant types in dp-minimal theories*, to appear in the J. Symbolic Logic.
- [6] ———, *On dp-minimal ordered structures*, *The Journal of Symbolic Logic*, vol. 76 (2011), no. 4, pp. 448–460.

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